Approximate solutions of the hybrid quantum Gowdy model with FRW dynamics

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Loops ′15, 9 July, 2015

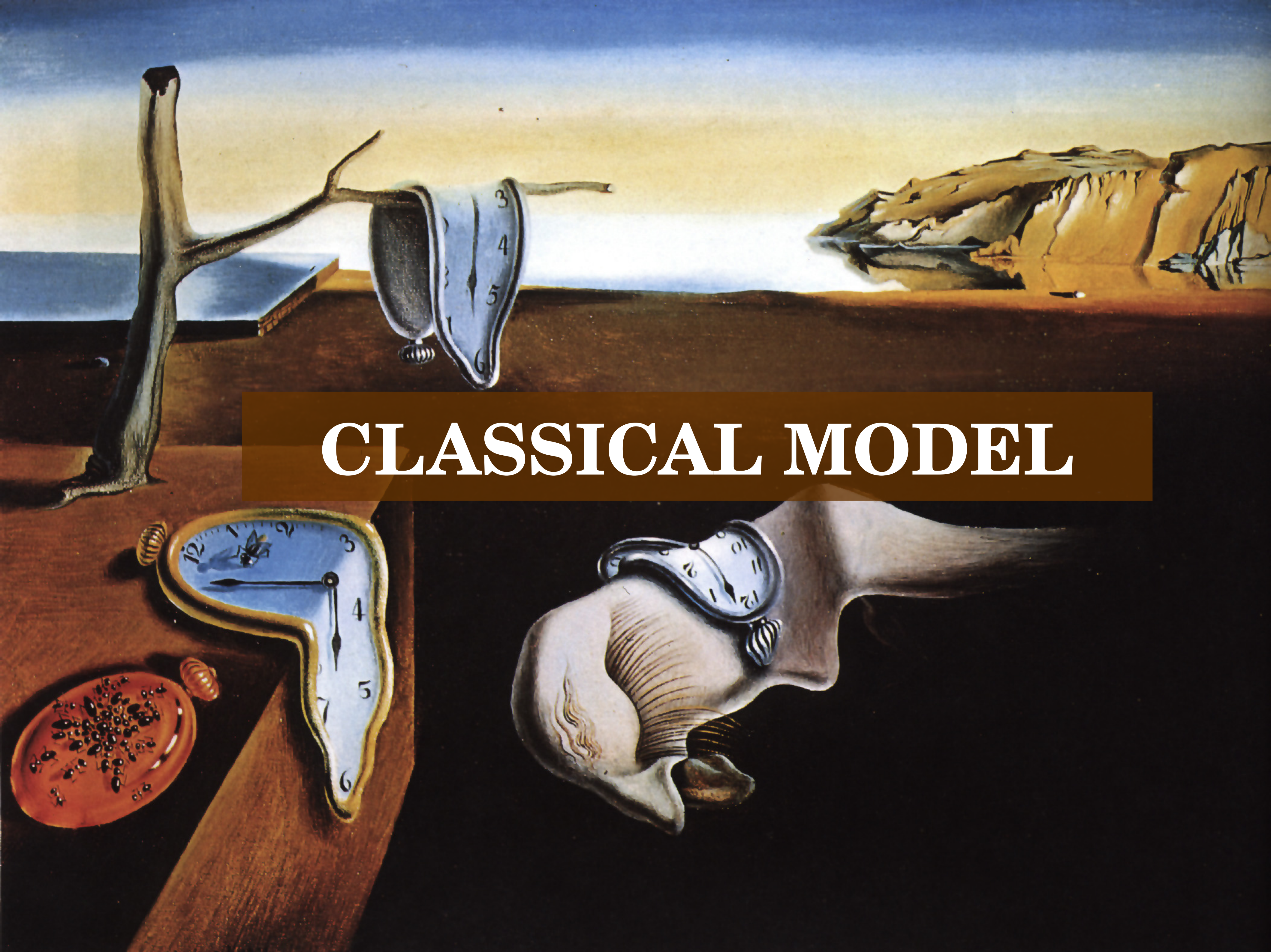
Overview

- Why LQC? Resolution of the Big Bang singularity (Big Bounce).

- Realistic scenarios require the study of inhomogeneous models, in this context: Gowdy cosmologies.

- Hybrid approach: loop quantization of the homogeneous background + Fock quantization of the inhomogeneous degrees of freedom.

- Approximation methods on the Hamiltonian constraint in order to find physical states.


CLASSICAL MODEL

Classical model: T^3 Gowdy

- Gravitational waves varying in one direction over a Bianchi I background.

- Linear polarization; (θ, σ, δ) orthogonal spatial angular coordinates.

- Two axial commuting Killing vectors $(\partial_\sigma, \partial_\delta)$.

- Inclusion of matter: Minimally coupled massless scalar field with the same symmetries: $\Phi = \Phi(\theta)$.

- Homogeneous sector with flat FRW solutions.

Reduced phase space

- Homogeneous sector:
  - Bianchi I with local rotational symmetry $(\sigma \leftrightarrow \delta)$.
  - Zero mode $\Phi_0 \equiv \phi$ & its momentum p_ϕ.

- Inhomogeneous sector:
  - Non-zero Fourier modes of the grav. wave $\xi(\theta)$ & its momentum.
  - Non-zero Fourier modes of $\Phi(\theta)$ & its momentum.

- Global constraints:

  Hamiltonian C_G, and momentum C_θ.

HYBRID QUANTIZATION

Homogeneous sector → Loop quantization

Inhomogeneous sector → Fock quantization

Hamiltonian Constraint Operator

$$\hat{C}_G = \underbrace{-\frac{3\kappa\hbar}{8}\hat{\Omega}^2 + \frac{\hat{p}_\phi^2}{2}}_{\hat{C}_{FRW}} \underbrace{-\frac{\kappa\hbar}{8}\left(\hat{\Theta}\hat{\Omega}+\hat{\Omega}\hat{\Theta}\right)}_{\hat{C}_{Ani}} + \underbrace{\frac{2\kappa\hbar}{\beta}\widehat{e^{2\Lambda}}\hat{H}_0}_{\hat{C}_0} + \underbrace{\frac{\kappa\hbar\beta}{4}\widehat{e^{-2\Lambda}}\hat{D}\hat{\Omega}^2\hat{D}\hat{H}_I}_{\hat{C}_I}$$

$\beta = const,\ \kappa \equiv \pi G \hbar$

$$Span\left\{|v\rangle\right\} \qquad v \in \left\{\varepsilon + 4n,\ n \in \mathbb{N}\right\},\ \varepsilon \in (0,4]$$

$$\langle v', \Lambda' | v, \Lambda\rangle = \delta_{v'v}\delta_{\Lambda'\Lambda} \qquad \otimes$$

$$Span\left\{|\Lambda\rangle\right\} \qquad \Lambda \in \left\{\text{countable dense set}\right\} \subset \mathbb{R}$$

$$\otimes$$

$$\mathscr{F}^{\xi} \qquad \xi \equiv \text{gravitational field}$$

$$\langle \boldsymbol{n}'_\xi, \boldsymbol{n}'_\varphi | \boldsymbol{n}_\xi, \boldsymbol{n}_\varphi\rangle = \delta_{\boldsymbol{n}'_\xi \boldsymbol{n}_\xi}\delta_{\boldsymbol{n}'_\varphi \boldsymbol{n}_\varphi} \qquad \otimes$$

$$\mathscr{F}^{\varphi} \qquad \varphi \equiv \text{rescaled matter field}$$

Hamiltonian Constraint Operator

$$\hat{C}_G = \underbrace{-\frac{3\kappa\hbar}{8}\hat{\Omega}^2 + \frac{\hat{p}_\phi^2}{2}}_{\hat{C}_{FRW}} \underbrace{-\frac{\kappa\hbar}{8}\left(\hat{\Theta}\hat{\Omega} + \hat{\Omega}\hat{\Theta}\right)}_{\hat{C}_{Ani}} \underbrace{+\frac{2\kappa\hbar}{\beta}\widehat{e^{2\Lambda}}\hat{H}_0}_{\hat{C}_0} \underbrace{+\frac{\kappa\hbar\beta}{4}\widehat{e^{-2\Lambda}}\hat{D}\hat{\Omega}^2\hat{D}\hat{H}_I}_{\hat{C}_I}$$

$\beta = const, \kappa \equiv \pi G\hbar$

- $\hat{\Omega}^2$ produces shifts of step 4 in v.
- $\left(\hat{\Theta}\hat{\Omega} + \hat{\Omega}\hat{\Theta}\right)$ produces shifts of step 4 in v and shifts in Λ that depend on v.
- $\hat{D}$ is $\hat{v}[\widehat{1/v}]$, does not commute with $\hat{\Omega}^2$.

Hamiltonian Constraint Operator

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$\beta = const\,,\ \kappa \equiv \pi G \hbar$

- $\hat{H}_0$ is the free field contribution, acts diagonally.
- $\hat{H}_I$ is the self-interaction, creates and annihilates particle pairs.

APPROXIMATIONS

Approximating $\hat{C}_{Ani}$

- Consider states $|g\rangle = \sum_{v,\Lambda} g(v,\Lambda)|v,\Lambda\rangle$, with $g(v,\Lambda)$ highly suppressed for $v \lesssim v_m \gg 10$.

contributing shifts not bigger than $q_\varepsilon = \log(1+2/v_m)$.

- If $g(v,\Lambda+\Lambda_0) \simeq g(v,\Lambda) + \Lambda_0 \partial_\Lambda g(v,\Lambda)$ for $\Lambda_0 \leq q_\varepsilon$:

$$\langle v,\Lambda|\hat{\Theta}\hat{\Omega} + \hat{\Omega}\hat{\Theta}|g\rangle \simeq -\langle v,\Lambda|2\hat{\tilde{\Omega}}\hat{\Theta}'|g\rangle,$$

$\hat{\Theta}'|\Lambda\rangle = i\dfrac{2}{q_\varepsilon}\Big(|\Lambda+q_\varepsilon\rangle - |\Lambda-q_\varepsilon\rangle\Big),$ $\hat{\tilde{\Omega}}$ shifts v in 4 units.

Disregarding $\hat{C}_{Ani}$

- Gaussian profiles peaked at $\bar{\Lambda}(\omega)$:

$$g(\omega,\Lambda)=N(\omega)\,e^{-\frac{\sigma_s^2}{2q_\varepsilon^2}[\Lambda-\bar{\Lambda}(\omega)]^2},$$

$\hat{\omega}$ essentially self-adjoint.

$$g(v,\Lambda)=\int_{Spc(\omega)} d\omega\, g(\omega,\Lambda)\, e_\omega(v)$$

suppressed for $v \lesssim v_m \gg 10$.

- If $q_\varepsilon \ll q_\varepsilon/\sigma_s \Leftrightarrow \sigma_s \ll 1$:

$$\langle v,\Lambda|\hat{\Theta}\hat{\Omega}+\hat{\Omega}\hat{\Theta}|g\rangle \simeq -\langle v,\Lambda|2\hat{\tilde{\Omega}}\hat{\Theta}'|g\rangle,$$

- If $\sigma_s \ll 1$:

$$\underbrace{\left|\langle v,\Lambda|\hat{\Theta}\hat{\Omega}+\hat{\Omega}\hat{\Theta}|g\rangle\right|}_{\hat{C}_{Ani}} \ll \underbrace{\left|\langle v,\Lambda|\hat{\Omega}^2|g\rangle\right|}_{\hat{C}_{FRW}} \qquad v \gtrsim v_m$$

Disregarding $\widehat{C}_I$

$$\widehat{C}_I \propto \widehat{e^{-2\Lambda}}\,\widehat{D}\,\widehat{\Omega}^2\,\widehat{D}\,\widehat{H}_I,$$

$$g(\omega,\Lambda)=N(\omega)\,\mathrm{e}^{-\frac{\sigma_s^2}{2q_\varepsilon^2}[\Lambda-\bar{\Lambda}(\omega)]^2},$$

$$\widehat{D}\left|v\right\rangle = D(v)\left|v\right\rangle \longrightarrow \left|v\right\rangle, v \gg 10$$

- On considered states, $\widehat{D}\,\widehat{\Omega}^2\,\widehat{D} \approx \widehat{\Omega}^2$.

- If $\bar{\Lambda}(\omega) \gg max(1, q_\varepsilon^2/\sigma_s^2)$:

$$\left|\left\langle v,\Lambda\right|\widehat{e^{-2\Lambda}}\,\widehat{\Omega}^2\left|g\right\rangle\right| \ll \left|\left\langle v,\Lambda\right|\widehat{\Omega}^2\left|g\right\rangle\right|,$$

  provided the content of inhomogeneities is reasonable.

- Recall $g(v,\Lambda)$ is highly suppressed for $v \lesssim v_m \gg 10$.

Approximating $\hat{C}_0$

$$\hat{C}_0 \propto \widehat{e^{2\Lambda}}\,\hat{H}_0\,, \qquad g(\omega,\Lambda) = N(\omega)\,e^{-\frac{\sigma_s^2}{2q_\varepsilon^2}[\Lambda-\bar{\Lambda}(\omega)]^2}\,,$$

- If

$$\frac{q_\varepsilon^2}{\sigma_s^2} \ll 1 \quad \Longrightarrow \quad \boxed{\widehat{e^{2\Lambda}}|g\rangle \approx e^{2\bar{\Lambda}(\hat{\omega})}|g\rangle}$$

- Consistency with $\sigma_s \ll 1$ requires:

$$\boxed{q_\varepsilon \ll \sigma_s \ll 1}$$

- This is compatible with the previous requirements:

$$\bar{\Lambda}(\omega) \gg 1 \gg q_\varepsilon^2/\sigma_s^2$$

Approximate Hamiltonian Constraint

$$\hat{C}_{app} = -\frac{3\kappa\hbar}{8}\hat{\Omega}^2 + \frac{\hat{p}_\phi^2}{2} + \frac{2\kappa\hbar}{\beta} e^{2\bar{\Lambda}(\hat{\omega})}\hat{H}_0$$

- Physical states

$$|\Psi\rangle = \int_{-\infty}^{\infty} dp_\phi \int_{Spc(\omega)} d\omega \sum_{\Lambda} \sum_{n^\xi, n^\varphi} \Psi(p_\phi, \omega, \Lambda, n^\xi, n^\varphi) \left| p_\phi, \omega, \Lambda, n^\xi, n^\varphi \right\rangle$$

must solve

$$\hat{C}_{app}|\Psi\rangle = 0$$

$$H_0 = H_0(n^\xi, n^\varphi)$$

$$\int_{Spc(\omega)} d\omega\, \Psi(p_\phi, \omega, \Lambda, n^\xi, n^\varphi) \left[-\frac{3\kappa\hbar}{8}\hat{\Omega}^2 + \frac{p_\phi^2}{2} + \frac{2\kappa\hbar}{\beta} e^{2\bar{\Lambda}(\hat{\omega})} H_0 \right] |\omega\rangle = 0$$

APPROXIMATE SOLUTIONS

Approximate solutions $\bar{\Lambda}(\omega)=\bar{\Lambda}_0$

$$\Psi(p_\phi, \omega, \Lambda, n^\xi, n^\varphi) = N(\omega, p_\phi, H_0)\, e^{-\frac{\sigma_s^2}{2q_\varepsilon^2}[\Lambda - \bar{\Lambda}(\omega)]^2}$$

- When $\bar{\Lambda}(\omega)=\bar{\Lambda}_0=const.$, the state factorizes:

$$\Psi(p_\phi, v, \Lambda, n^\xi, n^\varphi) = \psi(p_\phi, n^\xi, n^\varphi)\, N_{p_\phi, H_0}(v)\, f(\Lambda), \qquad f(\Lambda) = e^{-\frac{\sigma_s^2}{2q_\varepsilon^2}[\Lambda - \bar{\Lambda}_0]^2}$$

- Constraint equation $\longrightarrow$ eigenvalue equation:

$$\hat{\Omega}^2 N_{p_\phi, H_0}(v) = \left[\frac{4\, p_\phi^2}{3\kappa\hbar} + \frac{16}{3\beta} e^{2\bar{\Lambda}_0} H_0(n^\xi, n^\varphi)\right] N_{p_\phi, H_0}(v) \quad \Longleftrightarrow \quad \hat{\Omega}^2 e_\rho(v) = \rho^2 e_\rho(v)$$

- Important property of the eigenfunctions:

$e_\rho(v)$ exponentially suppressed for $\begin{cases} v \lesssim v_m = \rho/2 \\ \rho \gg 10 \end{cases}$

Approximate solutions $\bar{\Lambda}(\omega)=\bar{\Lambda}_0$

$$\Psi(p_\phi, v, \Lambda, n^\xi, n^\varphi)=\psi(p_\phi, n^\xi, n^\varphi)\, N_{p_\phi, H_0}(v)\, f(\Lambda), \qquad f(\Lambda)=e^{-\frac{\sigma_s^2}{2q_\varepsilon^2}[\Lambda-\bar{\Lambda}_0]^2}$$

$$N_{p_\phi, H_0}(v)=e_{\rho(p_\phi, H_0)}(v), \qquad \rho(p_\phi, H_0)=\sqrt{\frac{4}{3\kappa\hbar}p_\phi+\frac{16}{3\beta}e^{2\bar{\Lambda}_0}H_0}$$

- Exact solutions of $\hat{C}_{app}|\Psi\rangle=0.$
- Approximate ones of the Gowdy model if:
  - $q_\varepsilon \ll \sigma_s \ll 1, \bar{\Lambda}_0 \gg 1$
  - $v \gg 10 \longrightarrow \rho \gg 10 \longrightarrow p_\phi^2 \gg 75\kappa\hbar \approx 200 G\hbar^2$
  - $\hat{C}_\theta \psi(p_\phi, n^\xi, n^\varphi)=0$
- Effective constraint: FRW + massless scalar.

Approximate solutions $\bar{\Lambda}(\omega)$

$$\Psi(p_\phi, \omega, \Lambda, n^\xi, n^\varphi) = N(\omega, p_\phi, H_0)\, e^{-\frac{\sigma_s^2}{2q_\varepsilon^2}[\Lambda - \bar{\Lambda}(\omega)]^2} \qquad q_\varepsilon \ll \sigma_s \ll 1$$

$$\bar{\Lambda}(\hat{\omega}) = \begin{cases} \bar{\Lambda}_0, v < v_0 & \longrightarrow \text{Previous case, suppression for } v \lesssim v_m \\ \bar{\Lambda}(\hat{\omega}_0), v \geq v_0 & , v_0 \geq v_m \end{cases}$$

- $\hat{\omega}_0 = \hat{P}\hat{\omega}\hat{P}$, $\quad \hat{P}$ projector on $Span\{|v \geq v_0\rangle\}$
- $e^{2\bar{\Lambda}(\hat{\omega}_0)} = e^{2h(\hat{v})} + \hat{O}_p$, $\quad \hat{O}_p$ positive, defined on $Span\{|v \geq v_0\rangle\}$
- $\hat{O}_p$ with a quasi-local action that relates $2K < \infty$ volume states.

Approximate solutions $\bar{\Lambda}(\omega)$ $v \geq v_0$

$$\Psi(p_\phi, \omega, \Lambda, n^\xi, n^\varphi) = N(\omega, p_\phi, H_0)\, e^{-\frac{\sigma_s^2}{2q_\varepsilon^2}[\Lambda - \bar{\Lambda}(\omega)]^2}$$

$$N(v) \equiv \int_{Spc(\omega)} d\omega\, N(\omega, p_\phi, H_0)\, e_\omega(v)$$

- Constraint equation $\longrightarrow$ difference equation, relates:
  - 3 coefficients $N(v+4), N(v), N(v-4)$, from $\hat{\Omega}^2$.
  - K coefficients below v, from $\hat{O}_p$.
  - K coefficients above v, from $\hat{O}_p$.

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Unknown, not enough with the initial data

$$N(v_0 - 4) = e_{\rho(p_\phi, \Lambda(v_0), H_0)}(v_0 - 4),\ N(v_0).$$

Approximate solutions $\bar{\Lambda}(\omega)$ $v \geq v_0$

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  on at least v_0 and $2K-2$ points above.

  Determine up to $N(v_0+4K-4)$: K initial data!

Modified FRW $v \geq v_0$

- FRW+perfect fluid constraint:

$$\widehat{C}_{FRW+PF} = -\frac{3\kappa\hbar}{8}\widehat{\Omega}^2 + \frac{\widehat{p}_\phi^2}{2} + \alpha(1+w)\hat{v}^{1-w}, \quad p = w\epsilon$$

- Our constraint (solved by the previous states):

$$\widehat{C}_{app} = -\frac{3\kappa\hbar}{8}\widehat{\Omega}^2 + \frac{\widehat{p}_\phi^2}{2} + \frac{2\kappa\hbar}{\beta}\left[e^{2h(\hat{v})} + \widehat{O}_p\right]\widehat{H}_0$$

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$$h(v) = \frac{1}{2}\log\sum_w v^{(1-w)}, \quad h(v_0) \gg 1.$$

Dust $(w=0)$, radiation $(w=1/3)$,

cosmological const. $(w=-1)$.

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Geometrical corrections:

$$\left.\begin{array}{l}\text{Higher derivatives}\\[1ex]\text{Curvature-like terms}\end{array}\right\} f(R)\ldots$$

Conclusions

- Approximation methods to find quantum solutions for inhomogeneous cosmological models, in the context of LQC.
- States (anisotropy gaussians) which collectively behave as corresponding to an (unexpected) geometrically modified flat FRW model filled with perfect fluids, even when those states are genuinely anisotropic.
- The effective dynamics strongly depends on the considered family of states, and their quantum correlations.
- The developed approximation methods may be useful in the analysis of more realistic inhomogeneous scenarios.